1. Let $d < 0$ be a squarefree integer and let $F = \mathbf{Q}(\sqrt{d})$. Show that $O_F^* = \{-1, +1\}$ except when $d = -1$ or $d = -3$. Determine O_F^* also in those two cases.

2. Let $d \in \mathbf{Z}$ be positive but not a square. Show that Pell's equation

$$X^2 - dY^2 = 1$$

has infinitely many solutions $X, Y \in \mathbf{Z}$. (Hint: First deal with the case that d is the discriminant of a quadratic field).

3. Let F be a number field and let μ_F be the group of roots of unity in F. Let $\mathfrak{p}$ be a prime ideal of the ring of integers O_F and let p denote the characteristic of $O_F/\mathfrak{p}$. Show that the kernel of the reduction map $\mu_F \to (O_F/\mathfrak{p})^*$ is a p-group.

4. Let F be a number field. For a non-zero ideal $I \subset O_F$ we put $\Phi(I) = \#(O_F/I)^*$.
   (a) Show that $\sum_{I \subset J \subset R} \Phi(J) = N(I)$.
   (b) Show that

$$\Phi(I) = N(I) \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-1}).$$

   Here the product runs over the prime ideals $\mathfrak{p}$ with $I \subset \mathfrak{p} \subset O_F$.

5. We equip $\mathbf{R}^n$ with the usual inner product and let $L \subset \mathbf{R}^n$ be a lattice. The *dual lattice* L' is given by

$$L' = \{\mathbf{x} \in \mathbf{R}^n : \langle \mathbf{x}, \mathbf{y} \rangle \in \mathbf{Z} \text{ for all } \mathbf{y} \in L\}.$$

Show that L' is a lattice and show that $\mathrm{covol}(L') = 1/\mathrm{covol}(L)$.

6. (a) Show that the ring of integers O_F of $F = \mathbf{Q}(\sqrt[3]{20})$ is equal to $\mathbf{Z}[\sqrt[3]{20}, \sqrt[3]{50}]$.
   (b) Show that the discriminant of F is equal to -2700.
   (c) Let $a, b \in \mathbf{Z}$ and put $x = a\sqrt[3]{20} + b\sqrt[3]{50}$. Show that the discriminant $\Delta(1, x, x^2)$ is equal to $-2700(2a^3 - 5b^3)^2$.
   (d) Show that the Diophantine equation $2X^3 - 5Y^3 = \pm 1$ has no solutions $X, Y \in \mathbf{Z}$.
   (e) Show that there does not exist an element $\alpha \in O_F$ such that $O_F = \mathbf{Z}[\alpha]$.

7. Let $f(X) = X^3 - X^2 - 6X - 8 \in \mathbf{Z}[X]$ and let α denote a zero.
   (a) Show that f is irreducible.
   (b) Show that $\Delta(1, \alpha, \alpha^2) = -4 \cdot 431$. Show that the ring of integers O_F of $F = \mathbf{Q}(\alpha)$ admits $1, \alpha, \beta = (\alpha^2 - \alpha)/2$ as a $\mathbf{Z}$-basis.
   (c) Show that the kernel of the ring homomorphism $\mathbf{Z}[X, Y] \longrightarrow O_F$ that maps X to α and Y to β is equal to the ideal generated by $X^2 - X - 2Y$, $XY - 3X - 4$ and $Y^2 - 3Y + 2X - 2$).
   (d) Show that O_F has precisely three distinct ideals of index 2. Conclude that 2 splits completely in F.
   (e) Show that there is no $\alpha' \in F$ such that $O_F = \mathbf{Z}[\alpha']$. Show that for every $\alpha' \in O_F - \mathbf{Z}$, the prime 2 divides the index $[O_F : \mathbf{Z}[\alpha']]$.