1. Let $n > 1$ and $M \in \mathbf{R}$. Suppose that $x_1, \ldots, x_n \in \mathbf{R}$ satisfy

$$x_i \leq M \text{ for } i = 1, \ldots, n \qquad \text{and} \qquad \textstyle\sum_{i=1}^n x_i \geq 0.$$

   Prove the inequality $\sum_{i=1}^n x_i^2 \leq n(n-1)M^2$.

2. For $d = -11$, -21 and -23 compute the class group of the ring of integers O_F of $F = \mathbf{Q}(\sqrt{d})$.

3. Compute the class numbers of $\mathbf{Q}(\sqrt{-67})$ and of $\mathbf{Q}(\sqrt{-71})$.

4. For $d = 10$, 23 and 145 compute the class group and unit group of the ring of integers O_F of $F = \mathbf{Q}(\sqrt{d})$.

5. Compute the unit groups of $\mathbf{Q}(\sqrt{30})$ and of $\mathbf{Q}(\sqrt{31})$.

6. Let $p \equiv 3 \pmod 4$ be prime and let $F = \mathbf{Q}(\sqrt{-p})$. Show that the following are equivalent
   (a) the class number of F is 1;
   (b) $x^2 + x + \frac{p+1}{4}$ is prime for all integers x with $|x| < \frac{p-3}{4}$;
   (c) Check this with $p = 43$, $p = 67$ and $p = 163$.

7. Let p be a prime congruent to $1 \pmod 4$.
   (a) Prove that there exists $a \in \mathbf{Z}$ such that $a^2 \equiv -1 \pmod p$
   (b) Show that the covolume of L is equal to p, where

$$L = \{(x, y) \in \mathbf{Z}^2 : y \equiv ax \pmod p\}$$

   (e) Prove that every vector $(x, y) \in L$ satsfies $x^2 + y^2 \equiv 0 \pmod p$.
   (d) Let D be the disk D of center $(0, 0)$ and radius $\sqrt{2p}$. Prove that there is a non-zero vector (x, y) in $D \cap L$.
   (e) Prove that $p = x^2 + y^2$.

8. Let p be a prime number.
   (a) Prove that the intersection

$$\{-a^2 : a \in \mathbf{Z}/p\mathbf{Z}\}, \qquad \text{and} \qquad \{b^2 + 1 : b \in \mathbf{Z}/p\mathbf{Z}\}$$

   is not empty, so that there exist $a, b \in \mathbf{Z}$ with $-a^2 \equiv b^2 + 1 \pmod p$.
   (b) Prove that the lattice

$$L = \{(x, y, z, w) \in \mathbf{Z}^4 : z = ax + by \pmod p \text{ and } w = bx - ay \pmod p\}$$

   has covolume p^2 in $\mathbf{R}^4$.
   (c) Prove that every vector (x, y, z, w) in L satisfies $x^2 + y^2 + z^2 + w^2 \equiv 0 \pmod p$.
   (d) Show that the volume of the ball B in $\mathbf{R}^4$ of center $(0, 0, 0, 0)$ and radius $\sqrt{2p}$ is equal to $\frac{\pi^2}{2}(\sqrt{2p})^4$.
   (e) Prove that p is a sum of 4 squares.